On the Theory of Spectral Line Widths of Coupled Many Wave Systems in Non-Thermal Equilibrium

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The problem of the natural spectral line width $\Delta\omega$ of individual waves of a coupled many wave system in non-thermal equilibrium is discussed within the Fokker Planck approach to fluctuation phenomena. Edwards' method in turbulence theory is reinterpreted to relate $\Delta\omega$ via a modified energy balance equation to the spectrum of the mean wave intensities. This equation is solved for a model of the non-resonant feedback laser. $\Delta\omega$ is found to consist of two contributions: The first one is associated with the usual internal noise of a "free" Gaussian wave due to spontaneous emissions. Its contribution to the line width is twice that of a single mode laser far above threshold as found earlier by Brunner and Paul. The second and new contribution stems from the wave-wave coupling. It is similar to half the effective intensity fluctuation line width of the single mode laser and thus dominates above threshold.

1. Introduction

From the theory of the single mode laser it is known that the natural spectral line width of the emitted laser light is caused by spontaneous emission processes in the inverted atomic system.

The case of the single mode laser has been dealt with in detail using first principle methods of non-equilibrium quantum mechanics. These methods which established all statistical properties of the single mode laser are described by Haken [1]. The Fokker Planck (FP) approach to the statistics of laser light is expounded by Risken [2]. The Russian work on the natural spectral line width of lasers is reviewed by Klimontovich et al. [3].

The answer to the natural spectral line width $\Delta\omega$ of the single mode laser may be condensed into a simple formula

$$\Delta\omega = \alpha_L \, q / \langle I \rangle \, . \tag{1}$$

α_L is the line width factor which decreases monotonically from $\alpha_L = 2$ far below threshold to $\alpha_L = 1$ far above threshold. $\langle I \rangle$ is the mean light intensity and q the total strength of the spontaneous emission noise. A similar expression for $\Delta\omega$ appears already in early work on masers [4]. In fact, with the appropriate definition of q, Eq. (1) is the general result for the natural line width of a single self-sustained oscillation [5]. The physical problem is, of course, the determination of the functional dependence of α_L and q on the relevant parameters of the system. For the single mode laser this is accomplished in all details in the basic work reviewed in references given above.

For multi-mode operation no unique answer comparable to Eq. (1) is available. Actually, this cannot be expected because the result is bound to depend on the nature of the interaction between the modes, on the number of the modes and on the operating conditions. One therefore has to resort to model systems. Only a very few papers come up with explicit results on the spectral line width for special multi-wave systems. The general two mode laser is dealt with by Richter and Grossmann [6]. The intensity coupled many mode laser or non-resonant feedback laser (cf. [7]) is considered by Brunner and Paul [8] henceforth referred to as BP. The latter authors treat the coupled Heisenberg equations of motion including fluctuation operators (quantum Langevin equations [1]) of all the modes by heuristic averaging procedures of the non-linear feedback terms. They obtain in the threshold region Eq. (1) with $\alpha_L = 2$ and $\langle I \rangle$ = intensity of a single mode. This is the usual result for a linearly damped Gaussian process. As demonstrated by Ambartsumyan et al. [9] each individual mode is thermal. The result $\alpha_L = 2$ thus seems quite natural. However, in the BP approach, the fluctuation strength q is the same as for the corresponding single mode laser. Since any individual mode is coupled to an additional reservoir made up by the remaining $N-1$ modes, one expects a renormalization to occur in q, at least if the system is sufficiently far above threshold. This is the physical motivation for the present investigation.

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We concentrate on a model situation where $N \gg 1$ interacting modes are in non-thermal equilibrium. The interaction is assumed to be of a four wave (non-parametric) type which corresponds to a quartic interaction in the effective Hamiltonian [6]. We are forced to specialise the interaction in the course of the actual calculations. The full treatment of the problem will be given for a pure intensity coupling as described by Graham [10].

We use the non-resonant feedback laser, where the intensity coupling is of special simplicity, as our reference system for which explicit results will be derived. In detail, the course of the paper is as follows. In Sec. 2 the FP approach to the statistical problem of many interacting waves is discussed. In Sec. 3 we developed the basic concept of the present method based on ideas employed in turbulence theory and comment on its relation to the linear response formulation of Richter and Grossmann [6]. In Sec. 4, the first order calculation is developed for a typical four wave interaction. The general result is applied in Sec. 5 to the non-resonant feedback laser. In Sec. 6 the much more involved second order calculation is presented for intensity coupled waves. In Sec. 7 the second order calculation for the non-resonant feedback laser is checked against the result of Sec. 5 and discussed in detail. As expected from physical grounds, the wave-wave coupling turns out to be essential for the natural spectral line width above threshold.

A short communication of the present work has been given in [11].

2. Spectral Line Width and FP Approach

We are concerned with the spectral profile of an individual mode characterized by a mode index ν. The notion mode stands for an individual degree of freedom of the wave field under investigation. The spectral profile may be defined by the Fourier transform of the time correlation function

$$\langle \mathfrak{E}_\nu^{(-)}(t)\, \mathfrak{E}_\nu^{(+)} \rangle \,. \tag{2}$$

$\mathfrak{E}_\nu^{(+)}$ and $\mathfrak{E}_\nu^{(-)}$ are positive and negative frequency parts of the field operator associated with the mode ν. In [9] it is demonstrated how it is experimentally possible to investigate the correlation function of a single mode in a background of many other modes of a non-resonant feedback laser.

One way to determine the expression (2) is to make use of the quasi classical correspondence [1] and to set up the equation for the quasi distribution function $P(\{u_\nu\}t \mid \{u_\nu'\})$ for the complex c-number amplitudes $\{u_\nu\}$ associated with $\{\mathfrak{E}_\nu\}$. In a reasonable approximation, this equation reduces to an ordinary FP equation which may be written as ($\nu = \pm 1, \ldots, \pm N$)

$$\frac{\partial}{\partial t} P = -\sum_\nu \frac{\partial}{\partial u_\nu} [D_\nu(\{u_\mu\}) P] + 4 \sum_{\nu,\nu'} \frac{\partial^2}{\partial u_\nu\, \partial u_{\nu'}} (D_{\nu\nu'} P) \,. \tag{3}$$

The drift vector $D_\nu(\{u_\mu\})$ describes the mean motion and contains the interactions via non-linear terms. The diffusion matrix $D_{\nu\nu'}$ represents the macroscopic manifestations of fluctuation processes. In principle, D_ν and $D_{\nu\nu'}$ follow uniquely from microscopic calculations. In praxi, it is simpler to start from the coupled set of macroscopic equations of motion for the waves supplemented by fluctuating forces (Langevin equations) and to use the well known stochastic equivalence between these equations and a multi-dimensional FP equation [12].

The correlation function (2) then follows from the multi-dimensional integral

$$\int \prod_{\mu,\mu'} \mathrm{d}^2 u_\mu\, \mathrm{d}^2 u_{\mu'}'\, u_\nu\, u_\nu'^{*}\, P(\{u_\mu\}t \mid \{u_{\mu'}'\})\, W(\{u_{\mu'}'\}) \,.$$

Here P is the transition probability which is the fundamental solution of Equation (3). W is the stationary distribution function (assuming a physically stationary wave field) which follows from P in the limit $t \to \infty$. In a FP approach W as well as P are "true" probabilities, i. e. positive semi-definite functions.

The FP equation (3) provides a formal framework for the determination of all statistical properties of a coupled wave system. It must be said, however, that even on the basis of Eq. (3) analytic progress in our problem seems scarcely possible without involving methods similar to those of many body theory.

3. Basic Concepts

We consider a stationary system of many coupled waves in non-thermal equilibrium describable in terms of a FP equation. We adopt the exponential decay framework which states that perturbations in any wave will decay exponentially towards the equilibrium state, i. e. each wave is associated with just one decay constant. For sufficiently many interacting waves we expect the individual wave statistics to be Gaussian. Gaussian statistics in a Markoffian framework is necessarily associated with just one decay constant, namely the life time or equivalently the spectral line width of the wave. The exponential decay framework is an approximation which will be asymptotically correct for large numbers of interacting waves. It corresponds to the quasi particle hypothesis of statistical mechanics. It is expected not to be valid for a highly turbulent systems of waves [13]. The necessity of having many waves is demonstrated by the result for the single mode laser where more than one decay constant contribute to the amplitude and intensity fluctuations [2, 14, 15]. Within the exponential decay framework the FP description of the problem then is necessarily reduced to the following formal one. Replace the FP operator in Eq. (3) by a superimposition of Gaussian operators

$$\mathfrak{L}_{\mathrm{E}} = \sum_{\nu=-N}^{N} \mathfrak{L}_\nu , \tag{4}$$

where

$$\mathfrak{L}_\nu = \widetilde{D}_\nu (\partial/\partial u_\nu) u_\nu + \widetilde{Q}_\nu / 2 \, \partial^2/\partial u_\nu \, \partial u_{-\nu} .$$

$\widetilde{D}$ and $\widetilde{Q}$ are the renormalized linear damping and diffusion constants. For a damped Gaussian process, $\widetilde{D}$ is positive. Furthermore, $\widetilde{D}_\nu$, $\widetilde{Q}_\nu$ and the mean intensity are related by the basic relation

$$\widetilde{D}_\nu = \widetilde{Q}_\nu / (2 \langle u_\nu u_{-\nu} \rangle) . \tag{5}$$

The replacement has to be made in such a way as to preserve the moments for the stationary solutions of both equations. This is precisely the way, Edwards [16] attacks the turbulence problem which consists mainly in determining the moments $\langle u_k u_{-k} \rangle$ of the velocity field as function of wave vector k. In our non-turbulent case we assume these moments to be known and ask for $\widetilde{D}_\nu$ which is the required spectral line width of wave ν.

Before we go into the details of the calculation, we sketch briefly the renormalization procedure. It differs from the Edwards approach to turbulence because we deal with a trilinear interaction, i. e. the drift vectors are given by

$$D_\nu = D_{-\nu} = (D_\nu^{(0)} - \sum_{\nu' \nu''} M_{\nu \nu' \nu'' \nu} \, \dot{u}_{\nu'} u_{\nu''}) u_\nu . \tag{6}$$

M is a coupling matrix which obeys the reality condition

$$M_{-\nu \nu' \nu'' -\nu} = M^*_{\nu -\nu' -\nu'' \nu} . \tag{7}$$

A trilinear interaction instead of the parametric type bilinear one of turbulence is more appropriate for a many wave system like that of a laser where the wave frequencies are almost identical and the interaction comes about by absorption and emission processes between the waves [6]. This situation is brought about by external conditions (cavity) but mainly by the fixed atomic transition frequency.

In a schematic notation, the perturbation theory proceeds as follows. Since $\mathfrak{L} P = 0$ for the exact stationary distribution we have $(D_{\nu,\nu'} = Q_\nu \delta_{\nu,\nu'})$

$$\mathfrak{L}_{\mathrm{E}} P = (\mathfrak{L}_{\mathrm{E}} - \mathfrak{L}) P = (\widetilde{D} + D^{(0)})) \, \partial_u (u P) - M \, \partial_u (u u u P) + (\widetilde{Q} - Q)/2 \, \partial_u^2 P .$$

We now expand P around the Gaussian solution P_0 of $\mathfrak{L}_{\mathrm{E}}$:

$$P = P_0 + P_1 + P_2 + \dots .$$

We then have to assign orders to P_n, $\widetilde{D}$ and $\widetilde{Q}$ in terms of the coupling constant M. Clearly

$$P_n \propto M^n .$$

The situation for $\widetilde{D}$ and $\widetilde{Q}$ is more complicated. In contrast to the turbulence problem, where $\widetilde{D} + D^{(0)}$ and $(\widetilde{Q} - Q) = O(M^2)$, the only consistent assignment in our case is

$$\widetilde{D} = -D^{(0)} + O(M) , \tag{8'}$$

$$\widetilde{Q} = Q + O(M^2) . \tag{8''}$$

We thus get the perturbation sequence

$$\mathfrak{L}_{\mathrm{E}} P_0 = 0 , \tag{9'}$$

$$\mathfrak{L}_{\mathrm{E}} P_1 = (\widetilde{D} + D^{(0)}) \, \partial_u (u P_0) - M \, \partial_u (u u u P_0) , \tag{9''}$$

$$\mathfrak{L}_{\mathrm{E}} P_2 = (\widetilde{D} + D^{(0)}) \, \partial_u (u P_1) - M \, \partial_u (u u u P_1) + (\widetilde{Q} - Q)/2 \, \partial_u^2 P_0 . \tag{9'''}$$

These equations can sucessively be solved in terms of two-dimensional Hermite functions $H_{n_\nu n_{-\nu}}$ [16] which are polynomials in u_ν and $u_{-\nu}$ of order n_ν and $n_{-\nu}$ respectively times the Gaussian P_0 and obey

$$\mathfrak{L}_{\mathrm{E}} H_{n_\nu n_{-\nu}} = -\widetilde{D}_\nu (n_\nu + n_{-\nu}) H_{n_\nu n_{-\nu}} . \tag{10}$$

For any chosen order n in the coupling constants M, the integral equation involving $\widetilde{D}$ and $\widetilde{Q}$ follows by

$$\int \prod_\nu \mathrm{d}^2 u_\nu \, u_\mu \, u_{-\mu} \left(\sum_{m=1}^{n} P_m \right) = 0 \, ,$$

i. e. the condition, that already P_0 produces the correct moments $\langle u_\mu u_{-\mu} \rangle$. We shall work only to first and second order, since the higher order calculations increase enormously in complexity. A formal infinite order perturbation theory is given by Edwards [16] for the turbulence case.

Finally, we should like to comment on the relation of the present approach to the linear response treatment of Richter and Grossmann [6]. These authors perform a linear expansion of the $2N$ real fluctuation quantities x_i of a stationary coupled N wave system around the operating point (stationary state). This gives a set of $2N$ linear equations for $\{x_i\}$. The response function $\chi_{ij}(t)$ associated with this system of (second order) differential equations determines the spectral function $\varphi_{ij}(\omega)$ which is the Fourier transform of $\varphi_{ij}(t) = \langle x_i(t) x_j \rangle$ according to the dissipation fluctuation theorem

$$\varphi_{ij}(\omega) = 2\, q/\omega [(1/i) \operatorname{Im} \chi_{ij}(\omega)] \, . \qquad (11)$$

q is the "effective noise energy" of the wave system. Equation (11) clearly is the high temperature analogue of the Callen Welton theorem of equilibrium statistical mechanics. There is a rudimentary connection between the present treatment of a many wave system and the relation (11). For $N \to \infty$ and choosing $x_\nu = \operatorname{Re} u_\nu$, $x_{\nu+N} = 1/i \operatorname{Im} u_\nu$ only self correlations $i = j = \nu$ are expected to survive. Furthermore for a Lorentzian with width $\widetilde{D}_\nu$ integration of (11) over all frequencies ω gives

$$2\, \varphi_{\nu\nu}(0) = \langle u_\nu u_{-\nu} \rangle = 2\, q / \widetilde{D}_\nu \, .$$

This is almost Eq. (5) if one allows for a possible renormalization $q \to \widetilde{q}_\nu = \widetilde{Q}_\nu / 4$.

4. First Order Calculation

The first order calculation according to the description given above, turns out to be so simple that we can give it for the fairly general trilinear interaction according to Equation (6). Noting the conditions

$$\int \prod_\nu \mathrm{d}^2 u_\nu \, P_1 = 0 \, ,$$

$$\int \prod_\nu \mathrm{d}^2 u_\nu \, P_1 \, u_\mu \, u_{-\mu} = 0 \, ,$$

one gets

$$\int \prod_\nu \mathrm{d}^2 u_\nu \, u_\mu \, u_{-\mu} \, \mathfrak{L}_{\mathrm{E}} \, P_1 = 0 \, .$$

Using this equation in the first order formula (9″) and the conditions $\widetilde{D}_\nu = \widetilde{D}_{-\nu}$, $D_\nu^{(0)} = D_{-\nu}^{(0)}$ one obtains immediately

$$\begin{aligned} 2 (D_\mu + D_\mu^{(0)}) \langle u_\mu u_{-\mu} \rangle &= \sum_{\nu' \nu''} M_{\mu \nu' \nu'' \mu} \langle u_{\nu'} u_{\nu''} u_\mu u_{-\mu} \rangle_0 \\ &+ \sum_{\nu' \nu''} M_{-\mu \nu' \nu'' -\mu} \langle u_{\nu'} u_{\nu''} u_\mu u_{-\mu} \rangle_0 \, . \end{aligned}$$

$\langle \, \rangle_0$ denotes an average with respect to the Gaussian distribution P_0. Averages like this are evaluated by the usual contraction theorem which expresses higher order even moments by a sum of products over all possible pairs using

$$\langle u_\mu u_\nu \rangle_0 = \delta_{\mu, -\nu} \langle u_\mu u_{-\mu} \rangle \, .$$

Odd moments vanish as a consequence of the phase invariance in P_0. Using the symmetry property (7) of M one gets

$$\begin{aligned} \widetilde{D}_\mu = & -D_\mu^{(0)} + \tfrac{1}{2} \sum_\nu \langle u_\nu u_{-\nu} \rangle \qquad (12) \\ & \cdot [1 + \delta_{\nu,\mu} + \delta_{\nu,-\mu}] (\operatorname{Re} M_{\mu\nu-\nu\mu} + \operatorname{Re} M_{-\mu\nu-\nu-\mu}) \, . \end{aligned}$$

This is an explicit expression for the renormalized drift constants $\widetilde{D}$ in terms of the original linear drift coefficients $D^{(0)}$ and a weighted sum over the mode populations. For a linearly unstable system like a multi-mode laser above threshold, $D^{(0)} > 0$ while a stationary Gaussian process requires $\widetilde{D} > 0$. Consequently, the second term in (12) must not only be positive, but it must exceed $D^{(0)}$. It is further noted that Eq. (12) does not involve Q nor $\widetilde{Q}$, i. e. the fluctuation strengths of the original and the renormalized equations. This does not mean that there is no Q renormalization. In fact, comparing the result (12) for $\widetilde{D}$ with the basic relation (5) one usually will get $\widetilde{Q}$ values differing from the original Q's.

5. Model Calculation

The result Eq. (12) can easily be applied to the non-resonant feedback laser [7, 8]. Within a FP approach the essential features of the non-resonant feedback laser can be condensed into a model described and solved exactly in [17] for all static properties. In scaled notation the drift vector (6) follows from (cf also present Appendix)

$$D^{(0)} = N\,a\,,$$
$$M_{\mu\nu'\nu''\mu} = M\,\delta_{\nu',-\nu''} = 1/2\,\delta_{\nu',-\nu''}\,. \quad (13)$$

a is the pumping parameter, defined in analogy to the single mode laser case [2]. The static mean intensities are all equal: $\langle u_\mu u_{-\mu}\rangle \equiv \langle z\rangle$. From the exact expression for the mean intensity $\langle z\rangle$ one infers the relation

$$\langle z\rangle = a + 2/(N\langle z\rangle)\,, \quad (14)$$

which is valid for all a provided that $N \gg 1$. The renormalized drift constants according to Eq. (12) are identical for every wave and are given by

$$\widetilde{D} = (N+1)\langle z\rangle - N\,a\,.$$

Using the asymptotic expression (14) for $\langle z\rangle$ valid for $N \gg 1$, which is a necessary condition for the applicability of the method described here, we finally obtain:

$$\widetilde{D} \equiv \Delta\omega = a + 2/\langle z\rangle\,. \quad (15)$$

$\Delta\omega$ is the scaled spectral line width of each mode. As long as $a \ll 1$, the new result (15) is in accord with the BP formula $\alpha_{\mathrm{L}} = 2$. Far above threshold, $a \gg 1$, and Eq. (15) predicts a quite different behaviour of the spectral line width. A qualitatively new aspect of the result Eq. (15) for the spectral line width will be discussed in Sec. 6 where the unscaled expression for $\Delta\omega$ is worked out and the relation with the renormalized fluctuation strength is given.

6. Second Order Calculation

If the line width prediction Eq. (15) for the non-resonant feedback laser is to be reliable, this result should be obtained also from the second order calculation. This calculation is much more involved, since the function P_1 in Eq. (9″) has to be known explicitly. We therefore restrict ourselves to the case of real symmetric intensity interactions when M is given by

$$M_{\nu\nu'\nu''\nu} = \delta_{\nu',-\nu''}\,\beta_{\nu\nu'}\,,$$
$$\beta_{\nu\nu'} = \beta_{\nu'\nu} = \beta^*_{\nu\nu'}\,.$$

Such a model of randomly phased coupled waves is qualitatively discussed by Graham [Ref. [10], Ch. 6.2]. Quantum mechanical derivations are contained e. g. in [18, 19]. The special case $\beta_{\nu\nu'} = \mathrm{const.}$ is, of course, the reference model used her.

To proceed with the calculation, we work out the right hand side of Eq. (9″) using the explicit form for P_0:

$$P_0 = \prod_{\nu>0} (\pi\langle u_\nu u_{-\nu}\rangle)^{-1} \exp\left\{-\frac{u_\nu u_{-\nu}}{\langle u_\nu u_{-\nu}\rangle}\right\}, \quad (16)$$

which is properly normalized with respect to $\int\prod_{\nu>0} d^2u_\nu$. Performing the derivatives on the right hand side of Eq. (9″) gives:

$$\begin{aligned}\mathfrak{L}_{\mathrm{E}} P_1 = &-\prod_{\nu,\nu'} \beta_{\nu\nu'}[1+2\,\delta_{\nu,\nu'}]\;u_{\nu'}u_{-\nu'}\,P_0\\ &+\sum_{\nu,\nu'} \beta_{\nu\nu'}\,u_{\nu'}u_{-\nu'}\,\frac{u_\nu u_{-\nu}}{\langle u_\nu u_{-\nu}\rangle}\,P_0\\ &+\sum_{\nu} (\widetilde{D}_\nu + D_\nu^{(0)})\left(1-\frac{u_\nu u_{-\nu}}{\langle u_\nu u_{-\nu}\rangle}\right)P_0\,.\end{aligned} \quad (17)$$

To solve this equation for P_1, we have to expand the right hand side into two-dimensional Hermite functions. Since the second order calculation involves u_ν-terms up to forth order, only the first few Hermite functions are needed. Writing $H_{n_\nu n_{-\nu}} = \widetilde{H}_{n_\nu n_{-\nu}}\cdot P_0$ and adopting the scalar product

$$\langle \widetilde{H}_{n_\nu n_{-\nu}} | \widetilde{H}_{n_\mu n_{-\mu}}\rangle = \int\prod_{\varkappa>0} d^2u_\varkappa\, P_0\cdot \widetilde{H}^*_{n_\nu n_{-\nu}}\cdot \widetilde{H}_{n_\mu n_{-\mu}} = \delta_{n_\nu,n_\mu}\,\delta_{n_{-\nu},n_{-\mu}}\,\langle u_\nu u_{-\nu}\rangle^{(n_\nu+n_{-\nu})} \quad (18)$$

some of the first few Hermite polynomials $\widetilde{H}$ are explicity given by (using $u_\nu = u_{-\nu}^*$ and omitting the index ν):

$$\widetilde{H}_{00} = 1;\quad \widetilde{H}_{10} = u;\qquad \widetilde{H}_{20} = u^2/\sqrt{2}\,,$$
$$\widetilde{H}_{11} = u\,u^* - \langle u\,u^*\rangle\,,\qquad \widetilde{H}_{22} = u^2\,u^{*2}/2 - 2\,u\,u^*\langle u\,u^*\rangle + \langle u\,u^*\rangle^2\,.$$

Note that $\widetilde{H}^*_{n_{-\nu}n_\nu} = \widetilde{H}_{n_\nu n_{-\nu}}$. However, for intensity coupling, only those functions with $n_\nu = n_{-\nu}$ are needed which are real. The task to reorder the powers of u_ν in (17) such as to form Hermite polynomials is simple but tedious. The result is given by

$$\mathfrak{L}_{\mathrm{E}} P_1 = \Bigg[4 \sum_{\nu} \frac{\beta_{\nu\nu}}{\langle u_\nu u_{-\nu} \rangle} \widetilde{H}_{2\nu\, 2-\nu} + \sum_{\substack{\nu,\nu' \\ \nu \neq \pm \nu'}} \frac{\beta_{\nu\nu'}}{\langle u_\nu u_{-\nu} \rangle} \widetilde{H}_{1\nu\, 1-\nu} \widetilde{H}_{1\nu'\, 1\nu'}$$
$$+ \sum_{\nu,\nu'} \beta_{\nu\nu'} [1 + 2\,\delta_{\nu,\nu'}] \frac{\langle u_{\nu'} u_{-\nu'} \rangle}{\langle u_\nu u_{-\nu} \rangle} \widetilde{H}_{1\nu\, 1-\nu} - \sum_{\nu} \frac{\widetilde{D}_\nu + D_\nu^{(0)}}{\langle u_\nu u_{-\nu} \rangle} \widetilde{H}_{1\nu\, 1-\nu} \Bigg] P_0 . \tag{19}$$

Note that the second term in (19) which involves a product of Hermite polynomials is quasi linear due to $\nu \neq \pm \nu'$. We therefore can apply formula (10) to solve for P_1. The result which follows by inspection is:

$$P_1 = \Bigg[- \sum_{\nu} \frac{\beta_{\nu\nu}}{\langle u_\nu u_{-\nu} \rangle \widetilde{D}_\nu} \widetilde{H}_{2\nu\, 2-\nu} - \sum_{\substack{\nu,\nu' \\ \nu \neq \pm \nu'}} \frac{\beta_{\nu\nu'}}{2 \langle u_\nu u_{-\nu} \rangle (\widetilde{D}_\nu + \widetilde{D}_{\nu'})} \widetilde{H}_{1\nu\, 1-\nu} \widetilde{H}_{1\nu'\, 1-\nu'}$$
$$- \sum_{\nu,\nu'} \frac{\beta_{\nu\nu'} \langle u_{\nu'} u_{-\nu'} \rangle}{2 \widetilde{D}_\nu \langle u_\nu u_{-\nu} \rangle} [1 + 2\,\delta_{\nu,\nu'}] \widetilde{H}_{1\nu\, 1-\nu} + \sum_{\nu} \frac{\widetilde{D}_\nu + D_\nu^{(0)}}{2 \widetilde{D}_\nu \langle u_\nu u_{-\nu} \rangle} \widetilde{H}_{1\nu\, 1-\nu} \Bigg] P_0 . \tag{20}$$

The next steps are analogous to those of the first order calculation. We use

$$\int \prod_{\nu > 0} \mathrm{d}^2 u_\nu \, u_\mu u_{-\mu} (P_1 + P_2) = 0 ,$$

which leads to

$$\int \prod_{\nu > 0} \mathrm{d}^2 u_\nu \{ \mathfrak{L}_{\mathrm{E}} (P_1 + P_2) \} \, u_\mu u_{-\mu} = 0 ,$$

to obtain

$$0 = \sum_{\nu} \beta_{\mu\nu} \langle u_\nu u_{-\nu} u_\mu u_{-\mu} \rangle_0$$
$$- (\widetilde{D}_\mu + D_\mu^{(0)}) \langle u_\mu u_{-\mu} \rangle + \sum_{\nu} \beta_{\mu\nu} \langle u_\nu u_{-\nu} u_\mu u_{-\mu} \rangle_1 - (\widetilde{D}_\mu + D_\mu^{(0)}) \langle u_\mu u_{-\mu} \rangle_1 + (\widetilde{Q}_\mu - Q_\mu)/2 .$$

Averages $\langle \rangle_0$ are easily evaluated using the pairing theorem. $\langle \rangle_1$ denotes the more involved average taken with respect to P_1 These averages are best done by again reordering the powers of u into Hermite polynomials and making use of the scalar product Equation (18).

Proceeding along these lines a lengthy calculation finally gives the functional relation between the damping constants $\widetilde{D}_\nu$ and the intensity spectrum $\langle u_\nu u_{-\nu} \rangle \equiv \langle z_\nu \rangle$:

$$\frac{(\widetilde{D}_\mu + D_\mu^{(0)})^2}{\widetilde{D}_\mu} + D_\mu^{(0)} + \frac{Q_\mu}{2 \langle z_\mu \rangle} + 4 \langle z_\mu \rangle^2 \frac{\beta_{\mu\mu}^2}{\widetilde{D}_\mu} + 2 \sum_{\nu} \langle z_\nu \rangle \frac{\beta_{\mu\nu}^2}{\widetilde{D}_\nu + \widetilde{D}_\mu} (\langle z_\nu \rangle + \langle z_\mu \rangle)$$
$$+ \{ \sum_{\nu} \langle z_\nu \rangle \beta_{\mu\nu} [1 + 2\,\delta_{\mu,\nu}] \} \cdot \left\{ \sum_{\nu} \langle z_\nu \rangle \beta_{\mu\nu} [1 + 2\,\delta_{\mu,\nu}] \left(\frac{1}{\widetilde{D}_\mu} + \frac{1}{\widetilde{D}_\nu} \right) \right\} \tag{21}$$
$$= \sum_{\nu} \langle z_\nu \rangle \beta_{\mu\nu} [1 + 2\,\delta_{\mu,\nu}] \left(4 + \frac{D_\nu^{(0)}}{\widetilde{D}_\nu} + 2 \frac{D_\mu^{(0)}}{\widetilde{D}_\mu} \right).$$

This relation is the analogue of the energy balance equation in the Edwards theory of turbulence and is appropriate for our special trilinear intensity interaction in the drift vector.

A few points are noteworthy: The present method does not constitute an analytic expansion in β. In fact, for $\beta \to 0$ the zero-th order approximation namely the Gaussian P_0 according to Eq. (16) would be totally wrong. It is the finiteness of the mode coupling which makes P_0 a good zero-the order approximation. It is also pointed out that P_0 becomes a better and better approximation to the exact stationary solution the more variables are integrated out. Indeed any finite group of waves is expected to be asymptotically independent and Gaussian distributed.

7. The Spectral Line Width of the Non-Resonant Feedback Laser

We apply Eq. (21) to our model of the non-resonant feedback laser and get

$$(\tilde{D}+N a)^2+N a \tilde{D}+2 \tilde{D}/\langle z\rangle+(N+1)\langle z\rangle^2+2(N+1)^2\langle z\rangle^2=4 \tilde{D}(N+1)\langle z\rangle+3 N a(N+1)\langle z\rangle .$$

Introducing $x=(\tilde{D}+N a)/(\langle z\rangle(N+1))$ this equation can be written as

$$x^2-3 x+2=\frac{\tilde{D}(N+1)\langle z\rangle-N a \tilde{D}-2 \tilde{D}/\langle z\rangle-(N+1)\langle z\rangle^2}{(N+1)^2\langle z\rangle^2} .$$

As long as $\tilde{D}$ does not grow faster than $N^{2-\varepsilon}$, which would not make sense physically anyway, the r.h.s. vanishes in the limit $N\to\infty$. This gives the two solutions:

$$\tilde{D}_1=(N+1)\langle z\rangle-N a=a+2/\langle z\rangle , \quad \tilde{D}_2=2(N+1)\langle z\rangle-N a=(N+1)\langle z\rangle+D_1 .$$

The second solution is unbounded and will be discarded (cf. present Appendix). The first solution is the same as Eq. (15) obtained from the first order calculation. This provides the required consistency of our method.

We finally discuss the physical implications of the spectral line width formula Eq. (15) for the non-resonant feedback laser. In the first place we rescale this formula using the relation between pumping parameter a, strength Q of fluctuation force and the noise-free mode intensity $(u u^*)_{\text{op}}$

$$a=2\left[(u u^*)_{\text{op}} \Delta\alpha/Q\right]^{1/2} , \tag{22'}$$

as well as the relation between true and scaled frequencies

$$\omega_{\text{t}}=1/2\left[\Delta\alpha\, Q/(u u^*)_{\text{op}}\right]^{1/2} \omega_{\text{s}} , \tag{22''}$$

and true and scaled mean intensities

$$\langle u u^*\rangle_{\text{t}}=\langle I\rangle=1/2\left[(u u^*)_{\text{op}} Q/\Delta\alpha\right]^{1/2}\langle z\rangle_{\text{s}} , \tag{22'''}$$

to get

$$\Delta\omega=\Delta\alpha+Q/(2\langle I\rangle) . \tag{15'}$$

Obviously it is the quantity $\Delta\alpha$ which gives the difference to the BP result appropriate to an undressed Gaussian wave. It represents a typical many wave effect requiring $N \gg 1$ coupled waves. While for a single wave system in the rotating wave van der Pol description $\Delta\alpha$ would be the linear gain, $\Delta\alpha>0$ represents the differential loss per wave in the case of a many wave system fluctuating around an operating point far from thermal equilibrium. $\Delta\alpha$ then is predominantly determined by the interactions between the waves and differs totally from any linear gain. Explicit examples have been given by BP for the non-resonant feedback laser near threshold (cf. present Appendix) when trivially $\Delta\alpha=\alpha_{\text{lin}}/N$ ($\alpha_{\text{lin}}=$ linear gain) and by Wonneberger et al. [20] for the case of highly amplified vibrational noise arbitrarily far above threshold. From its meaning of the differential loss it is clear that $\Delta\alpha$ adds to the internal noise part $Q/(2\langle I\rangle)$ of the spectral line width of an undressed Gaussian wave. Note that $\Delta\alpha<0$ corresponds to a wave system below threshold which is simulated by $a<0$. From the dependence of $\langle z\rangle$ on a it follows that $\Delta\alpha$ plays no role in Eq. (15′) under this condition. The major physical difference to the result for the single wave system is, that $\Delta\omega$ remains finite even if the internal noise Q vanishes. It is now the mode coupling which makes the remaining waves act as a reservoir and consequently as a noise source on each individual mode. This effect dominates the internal noise for $a \gtrsim 1$. This becomes particularly clear if one introduces the line width factor according to

$$\alpha_{\text{L}}=\Delta\omega/\left[Q/(4\langle I\rangle)\right] , \tag{23}$$

which is twice the ratio of the actual spectral line width to that of an undressed thermal wave. Inserting Eq. (15′) and using the definitions (22′), (22″) one gets

$$\alpha_{\text{L}}=2+a\langle z\rangle . \tag{24}$$

Here, 2 corresponds to the BP result and $a\langle z\rangle$ describes the mode coupling noise. Under the conditions $a>0$, $N a^2 \gg 1$ this formula simplifies to

$$\alpha_{\text{L}}=2+a^2 . \tag{24'}$$

In terms of the Edwards method the change in the line width is brought about by the renormalization of the strength Q of the fluctuating forces: $Q\to\tilde{Q}$. Evidently this normalization is described by

$$\tilde{Q}=1/2\, \alpha_{\text{L}}\, Q .$$

It is interesting to compare $\Delta\omega_I^{(N)} \equiv 2\,\Delta\omega$ which is the intensity fluctuation line width for a Gaussian wave with Lorentzian spectrum with the corresponding expression arising in the theory of the single mode laser. Though it is known that in this case the intensity fluctuation spectrum is not described by just one decay constant, there exists to a reasonable degree of accuracy an effective line width [21] given by

$$\Delta\omega_{\mathrm{I}} = a\,\lambda_{\mathrm{eff}}(a)\,Q/[4(u\,u^*)_{\mathrm{op}}]\,.$$

The effective eigenvalue $\lambda_{\mathrm{eff}}(a)$ is tabulated in [22]. Far above threshold $\lambda_{\mathrm{eff}}(a)$ takes the value $2\,a$ in which case one obtains $\Delta\omega_{\mathrm{I}} = 2\,\Delta a$. Again, Δa is the linear gain of the single mode laser. In the entire operation range and in scaled notation one has to compare $\lambda_{\mathrm{eff}}(a)$ with

$$\lambda^{(N)}(\mathrm{a}) = 2\,a + 4/\langle z\rangle\,. \tag{25}$$

Using the exact result for $\langle z\rangle$ [17] some characteristic values of $\lambda^{(N)}(a)$ are summarized by

$$\lambda^{(N)}(a) = \begin{cases} 2\,a + 4/a - 8/(N\,a^3)\,; & a>0\,,\quad N\,a^2 \gg 1\,, \\ 2\,\sqrt{2\,N}\,; & a=0\,, \\ 2\,(N-1)\,|a|\,; & a<0\,,\quad N\,a^2 \gg 1\,. \end{cases}$$

In all cases it is required that $N \gg 1$ which in praxi means $N \gtrsim 5$. As has been stressed earlier the many wave nature has been put into the theory from the start on by adopting a turbulence like description. It is seen that $\lambda^{(N)}(a)$ like $\lambda_{\mathrm{eff}}(a)$ is unsymmetric with respect to a. Similarily, $\lambda^{(N)}(a)$ diverges for $|a| \to \infty$. The minimum value is near ($a = \sqrt{2}$; $\lambda = 4\,\sqrt{2}$) which is the precise location of the minimum for $N \to \infty$. In this limit a singularity in λ appears at $a = 0$. This is a simple example of a dynamical critical behaviour of a cooperative system of infinitely many degrees of freedom.

8. Discussion

We have investigated the problem of the natural spectral line width of an individual wave which is a member of a system of many coupled waves in non-thermal equilibrium. Such a situation arises when a non-linear amplification mechanism produces a steady state output having many statistical degrees of freedom. The problem has been attacked using the Fokker Planck approach to fluctuation phenomena. The physically as well as mathematically related Edwards' method in stationary turbulence theory has been found to be applicable to the spectral line width problem. This method aims at substituting a renormalized equation for "free" Gaussian waves for the FP equation for the many coupled waves. The damping constants appearing in this equation have been identified as the proper spectral line widths for well behaved (non-turbulent) wave systems. For known mean wave intensities the spectral line widths can be determined explicitly by the Edwards' expansion procedure. First and second order calculations in the wave-wave coupling constants have been given. The first order calculation directly expresses the line width as a functional of the mean wave intensities. The second order equation has been derived for pure intensity coupling. It is the equivalent to the energy balance equation in turbulence theory and consequently an involved non-linear integral equation. Both equations have been applied to a model for the non-resonant feedback laser for which the static mean intensities are all equal and can be computed exactly. The result is found to be identical in both calculations. Apart from the usual Gaussian noise contribution there is a term due to the wave-wave coupling which dominates above threshold.

The result can also be understood in terms of a renormalized strength of the fluctuating force acting on any individual wave due to coupling this wave to a reservoir made up by the rest of the wave system.

The applied method rests on three suppositions:

a) All waves in a small subgroup of waves of a coupled many wave system are statistically thermal and independent.
b) The quasi "particle" hypothesis applies, i. e. the damping is small and exponential.
c) The mean wave intensities ("spectrum") are known precisely.

For the example given, conditions a) and c) can be shown to hold. Condition b) remains to be justified though it is physically appealing and in accord with the FP approach.

It remains to be seen whether more advanced model solutions to the problem of many coupled waves in non-thermal equilibrium can be found.

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Appendix

In the main text the non-resonant feedback laser has been discussed within an abstract model involving the pumping parameter and scaling relations involving the differential loss $\Delta\alpha$, the noise free mode intensity $(u\,u^*)_{\text{op}}$ and the strength Q of the fluctuating forces. In this appendix we relate these quantities to the microscopic parameters of the BP theory of the non-resonant feedback laser near threshold.

The gain operator T′ in BP corresponds to the expression for the gain α in [17] i. e.

$$T' = -\varkappa + \frac{N_{\text{A}}\,h^2\,\sigma_0}{\Gamma}\left(1 - \frac{4\,h^2}{\Gamma^2}\sum_{\lambda} b_\lambda^+\,b_\lambda\right), \quad \text{(A1')}$$

$$\mathrel{\hat{=}} \Delta\alpha\left\{\sum_{\nu'}\left(1 - \frac{u_{\nu'}^*\,u_{\nu'}}{(u^*\,u)_{\text{op}}}\right)\right\}. \quad \text{(A1'')}$$

This gives by mere comparison

$$\Delta\alpha = \varkappa(\sigma_N - 1)/N\,,$$
$$(u\,u^*)_{\text{op}} = \frac{\Gamma^2}{4\,h^2}\,\frac{1}{N}\,\frac{\sigma_N - 1}{\sigma_N}\,, \quad \text{(A2)}$$
$$Q = 2\,\frac{N_{\text{A}}\,h^2\,\beta}{\Gamma^2}\,\sigma_N\,.$$

In these equations the following definitions are used: N_{A} = number of active atoms, $\varkappa$ = cavity loss constant, h = coupling constant between atom and radiation field, $\Gamma = \beta + \gamma$ = (homogeneous) line width of the atoms = sum of diagonal and non-diagonal relaxation rates, $\sigma_N = N_{\text{A}}\,h^2\,\sigma_0/(\Gamma\varkappa)$ = normalized pump strength ($\sigma_N \approx 1$ in the BP theory), σ_0 = field free pump inversion per atom. The pumping parameter is explicitly given by

$$a = \frac{1}{N}\,\frac{\sigma_N - 1}{\sigma_N}\,\frac{\Gamma^2}{h^2}\left[\frac{\varkappa}{2\,N_{\text{A}}\,\beta}\right]^{1/2}. \quad \text{(A3)}$$

These formulae establish a one to one correspondence between our model system and the BP theory of the non-resonant feedback laser. They are valid in a lasing region near threshold when the number of induced photons exceeds the number of spontaneous photons. BP also discuss the case when very near threshold the laser is flooded by spontaneous photons in many modes. Under this condition they obtain an expression for the line width equivalent to

$$\Delta\omega = N\,\Delta\alpha\,. \quad \text{(A4)}$$

This is formally the linear gain associated with Eq. (A1′) and essentially the second solution of our Eq. (21) which we have discarded. We are allowed to do this since this situation does not correspond to a true multi-mode operation which is inherent in our model from the start on by fixing the number N. Furthermore, the physically new aspects of our formula Eq. (15) appear for $a \gtrsim 1$, i. e. outside the threshold region where Eq. (A4) is not applicable.

[1] H. Haken, Laser Theory, in: Encyclopedia of Physics, Vol. XXV/2c, ed. S. Flügge, Springer-Verlag, Berlin 1970.

[2] H. Risken, in: Progress in Optics, Vol. VIII, ed. E. Wolf, North-Holland Publishing Co., Amsterdam 1970, p. 239.

[3] Yu. L. Klimontovich, A. S. Kovalev, and P. S. Landa, Usp. Fiz. Nauk **106**, 279 [1972] (Sov. Phys. Usp. **15**, 95 [1972]).

[4] J. P. Gordon, H. J. Zeiger, and C. H. Townes, Phys. Rev. **99**, 1264 [1955].

[5] R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 [1967].

[6] P. H. Richter and S. Grossmann, Z. Phys. **255**, 59 [1972].

[7] R. V. Ambartsumyan, N. G. Basov, P. G. Kryukov, and V. S. Letokhov, in: Prog. Quant. Electr. Vol. I/3, eds. J. H. Sanders and K. W. H. Stevens, Pergamon Press, Oxford 1969, p. 107.

[8] W. Brunner and H. Paul, Ann. Physik **23**, 152 and 384 [1969].

[9] R. V. Ambartsumyan, P. G. Kryukov, V. S. Letokhov, and Yu. A. Matveets, Zh. Eksp. Teor. Fiz. **53**, 1955 [1967] (Sov. Phys. JETP **26**, 1109 [1967]).

[10] R. Graham, Springer Tracts in Modern Physics, Vol. 66, Springer-Verlag, Berlin 1973, p. 1.

[11] W. Wonneberger. J. Phys. A Letters: Gen. Phys. **7**, No. 11 [1974].

[12] R. L. Stratanovich, Topics in the Theory of Random Noise, Gordon and Breach, New York 1963.

[13] S. F. Edwards and W. D. McComb, J. Phys. A: Gen. Phys. **2**, 157 [1969].

[14] S. Chopra and L. Mandel, IEEE J. Quantum Electr. **8**, 324 [1972].

[15] M. Corti, V. Degiorgio, and F. T. Areçchi, Optics Commun. **8**, 329 [1973].

[16] S. F. Edwards, J. Fluid Mech. **18**, 239 [1964].

[17] G. Rowlands and W. Wonneberger, to be published in J. Phys. A: Gen. Phys., and W. Wonneberger and J. Lempert, Z. Naturforsch. **28 a**, 762 [1973].

[18] H. Haken, Z. Physik **265**, 105 [1973].

[19] H. Dekker, Optics Commun. **10**, 114 [1974].

[20] W. Wonneberger, J. Lempert, and W. Wettling, J. Phys. C: Solid St. Phys. **7**, 1428 [1974].

[21] H. Risken and H. D. Vollmer, Z. Physik **201**, 329 [1967].

[22] H. Risken, Fortschr. Physik **16**, 261 [1968].